A STUDY ON GAUSSIAN TWO-USER CHANNELS

BY

HIROSHI SATO

JANUARY 1976

TECHNICAL REPORT B76-2

A STUDY ON GAUSSIAN TWO-USER CHANNELS

by

Hiroshi Sato
University of Hawaii
Honolulu, Hawaii

January 1976

Sponsored by
Advanced Research Projects Agency
ARPA Order No. 1956

*****

A STUDY ON GAUSSIAN TWO-USER CHANNELS

by

Hiroshi Sato*

Abstract

In this report, we study the Gaussian two-user channel which is an extension of an ordinary Gaussian channel with single input and output to the one with two inputs and two outputs and where the independent informations are transmitted between two input/output pairs simultaneously. Two achievable regions called TS and FDM1 are compared. TS is obtained by simply time-sharing the two superposition modes introduced in the previous paper and FDM1 corresponds to the frequency division multiplexing where only one of the two signals is transmitted in each of the two sub-bands. A fairly tight outer bound to the achievable region is obtained by utilizing the capacity region of the corresponding Gaussian broadcast channel. Properties of two kinds of frequency division multiplexing FDM2 and FDM3 which make use of the superposition modes are studied and the achievability attained by these two kinds of multiplexing is discussed using the results of computation.

This report was supported by THE ALOHA SYSTEM, a research project at the University of Hawaii, which is supported by the Advanced Research Projects Agency of the Department of Defense and monitored by NASA Ames Research Center under Contract No. NAS2-8590.

* *Dr. Sato is on leave with THE ALOHA SYSTEM at the University of Hawaii from the University of Electro-Communications, Chofu, Tokyo, Japan, for one year.*

I. INTRODUCTION

In a previous paper [1] called "Two-User Communication Channels," we discussed in some detail the achievable region of rate pairs of "two-user channels." By "two-user channels," we mean the channels with two inputs and two outputs. We discussed the achievability of rate pairs when separate messages are sent between two source-user pairs. A "Gaussian two-user channel" is a typical example of the continuous version of this class of channels and is an extension of an ordinary Gaussian channel with single input and output to the one with two inputs and two outputs. This channel was studied in that paper to some extent giving us much insight into the general two-user channel problem. It seems, however, that some problems proper to the Gaussian channel were left unsolved, e.g., whether some kinds of time or frequency division multiplexing can further improve the achievable region? or how is the problem related to that of the Gaussian broadcast channel studied thoroughly by Bergmans and Cover [2]?

The purpose of this paper is to give some answers to the above questions. Although we cannot yet obtain the exact capacity region of this channel, we succeed in obtaining a tighter outer bound by utilizing the known exact capacity region of the corresponding Gaussian broadcast channel. We also improve on the achievable region obtained in the previous paper by introducing two kinds of frequency division multiplexing.

In Section II, the concept of Gaussian two-user channels is explained. Two kinds of superposition modes and the region of the achievable rate pairs obtained by time-sharing the above two superposition modes are reviewed. Another achievable region obtained by simple time or frequency division multiplexing mentioned in the previous paper and an outer bound to the

achievable region obtained in that paper are also reviewed briefly. In Section III it is shown that the exact capacity region for the corresponding broadcast channel problem can be utilized as a tighter outer bound to our problem than that introduced in the previous paper. In Section IV, two achievable regions introduced in the previous paper and reviewed in Section II are compared in many situations. In Section V we study frequency division of two superposition modes where a given frequency band is divided into two sub-bands and signals are transmitted by mode 1 and mode 2 within the respective bands. In Section VI frequency band is also divided into two but signals are transmitted by the same mode within both bands. It is shown that these kinds of multiplexing can somewhat improve the achievable rates and in Section VII their performance is discussed using the results of numerical calculation.

II. SUPERPOSITION MODES AND SIMPLE FREQUENCY DIVISION MULTIPLEXING (FDM1)

Let two radio transmitters independently transmit continuous signals $x_1(t)$ and $x_2(t)$ with allotted powers P_1 and P_2 and both with the same available bandwidth W to two receivers 1 and 2 respectively. As is shown in Fig. 1 the two signals are superposed first and then the sum $x(t)=x_1(t)+x_2(t)$ is sent to receiver 1 and 2 in the presence of additive white Gaussian noises $Z_1(t)$ and $Z_2(t)$ of one-sided power spectral density N_1 and N_2 respectively. We assume $N_1 \leq N_2$ throughout this paper. This channel as a whole is called a Gaussian two-user channel. We assume also that separate messages are sent between two

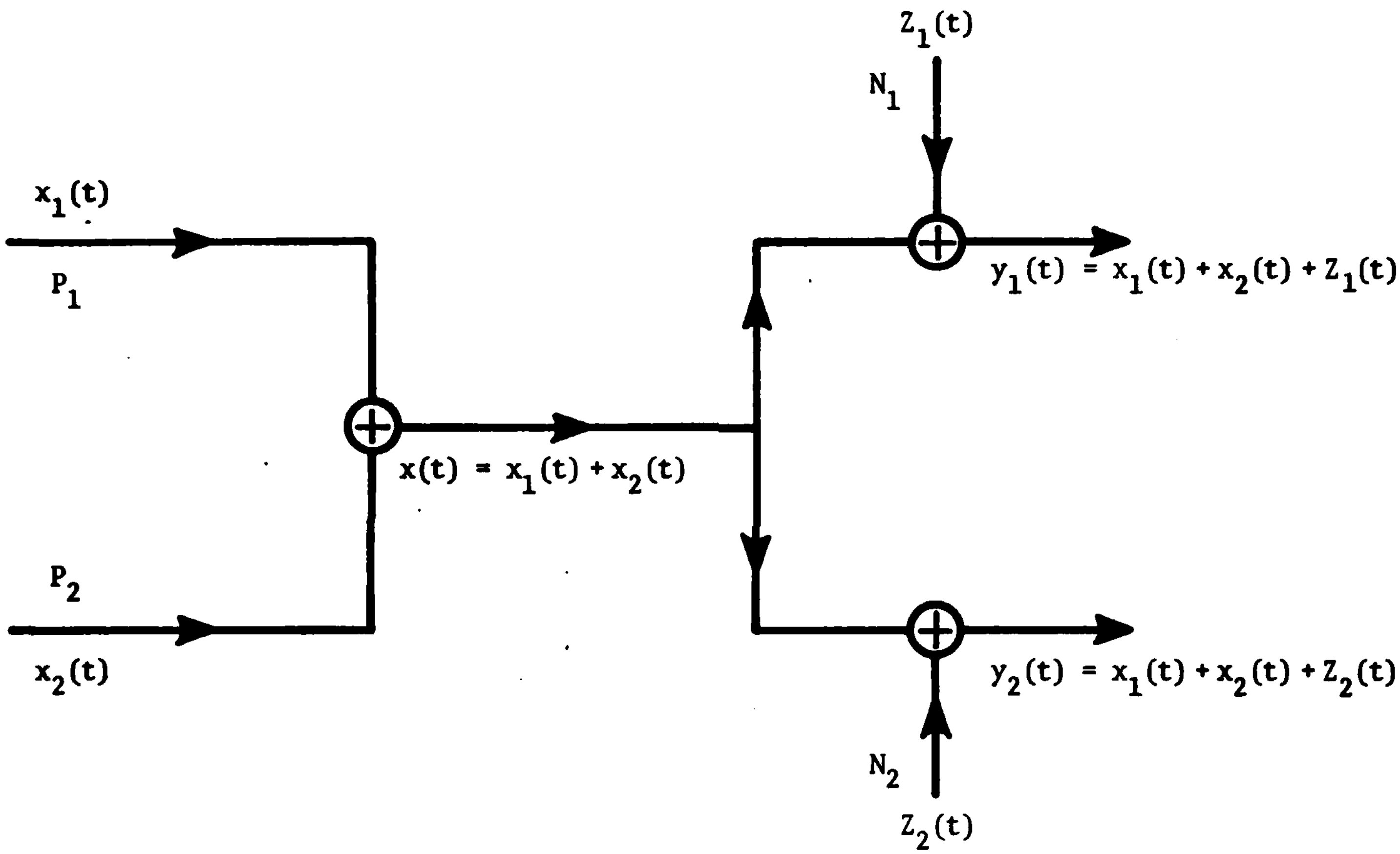


Figure 1

GAUSSIAN TWO-USER CHANNEL

input/output pairs as in the previous paper. Then we ask what set of rates (R_1,R_2) are simultaneously achievable in this channel.

First let a deterministic signal $x_2(t)$ of power P_2 be sent through the channel, then we can transmit signal $x_1(t)$ with maximum rate

$$C_1^0 = W\log\left(1+\frac{P_1}{WN_1}\right) , \tag{1}$$

(let us take 2 as a basis of logarithm throughout this paper) because receiver 1 can first subtract the known signal $x_2(t)$ from the received signal $y_1(t)$ and then decode $x_1(t)$ as in the single input/output channel. Therefore we can see that the rate pair $(C_1^0,0)$ is achievable. In the same way, we can see that the rate pair $(0,C_2^0)$ is also achievable, where

$$C_2^0 = W\log\left(1+\frac{P_2}{WN_2}\right) . \tag{2}$$

In a previous paper, we showed that the rate pairs (C_1^0,C_2) and (C_{12},C_2^0) are also achievable, where

$$C_2 = W\log\left(1+\frac{P_2}{P_1+WN_2}\right) , \tag{3}$$

and

$$C_{12} = W\log\left(1+\frac{P_1}{P_2+WN_2}\right) . \tag{4}$$

We first explain the achievability of the first rate pair. On the one hand, receiver 2 can decode code words with rate C_2 in (3) with as small an error as

possible because $x_1(t)$ with power P_1 is regarded as noise in addition to $n_2(t)$ with power WN_2 when we cannot know what $x_1(t)$ is being sent. On the other hand, receiver 1 can first decode x_2 correctly. It is possible because he can in general decode $x_2(t)$ of power P_2 in the presence of signal $x_1(t)$ of power P_1 and a noise of power WN_1 when the transmission rate of the second message R_2 does not exceed

$$W\log\left(1+\frac{P_2}{P_1+WN_1}\right),$$

and this is in fact larger than C_2 in (3) because of the inequality $N_1 \leq N_2$. As a next step, receiver 1 can decode a signal $x_1(t)$ with rate as large as C_1^0 in (1) from the signal $x_1(t)+n_1(t)$ obtained by subtracting the already decoded $x_2(t)$ from received signal $y_1(t)$. In this way the rate pair (C_1^0, C_2) has been shown achievable. Let us call the above-mentioned mode of communication as superposition mode 1. Next, the achievability of the second rate pair (C_{12}, C_2^0) is explained in the following fashion. First both receivers can decode $x_1(t)$ correctly from the respective received signals. This would be clear to receiver 2 from the functional form of C_{12} in (4), and is also true to receiver 1 because of the inequality $N_1 \leq N_2$. Receiver 2 can then decode $x_2(t)$ of rate C_2^0 in (2) after subtracting the already decoded $x_1(t)$ from the received signal. Thus the rate pair (C_{12}, C_2^0) has been shown to be achievable. Let us call this mode of communication as superposition mode 2. It should be noted here that these two modes are not symmetrical because of the restriction $N_1 \leq N_2$. One characteristic feature of mode 2 is that two independent messages sent from two inputs can both be decoded correctly at each of two outputs.

Let us consider a rectangle in a R_1-R_2 plane surrounded by straight lines $R_1=C_1^0$, $R_2=C_2^0$, R_1-axis and R_2-axis, then the rate pairs (C_1^0, C_2) and (C_{12}, C_2^0) are

represented by points A_1 and A_2 on the sides of the rectangle as shown in Fig. 2(a). If we connect A_1 and A_2 by a straight line, then the rate pair corresponding to the point on this line between A_1 and A_2 can be easily shown to be achievable by the so-called "time sharing" argument. Suppose this two-user channel is used for a sufficiently long period of time T. If in a part λT of the total period the channel is operated by the superposition mode 1 and in the remaining $(1-\lambda)T$ of the period it is operated by the mode 2 for a certain λ, $0 \leq \lambda \leq 1$, then we can get a rate pair represented by the following equation and the corresponding point is on the straight line segment $\overline{A_1A_2}$:

$$\begin{aligned} R_1 &= \lambda C_1^0 + \bar{\lambda} C_{12} \\ R_2 &= \lambda C_2 + \bar{\lambda} C_2^0 \end{aligned} \tag{5}$$

where

$$0 \leq \lambda \leq 1, \qquad \bar{\lambda} = 1 - \lambda \ .$$

Let us denote this line segment $\overline{A_1A_2}$ by TS. The part of the rectangular region on the left-lower side of this TS is an achievable region, and we sometimes call this achievable region TS. It is shown in Section IV that this time-sharing rate pair on $\overline{A_1A_2}$ is also realized by a frequency division multiplexing.

In the previous paper, we noted that by a simple time division multiplexing (let us call this TDM1) the following rate pair can be achieved:

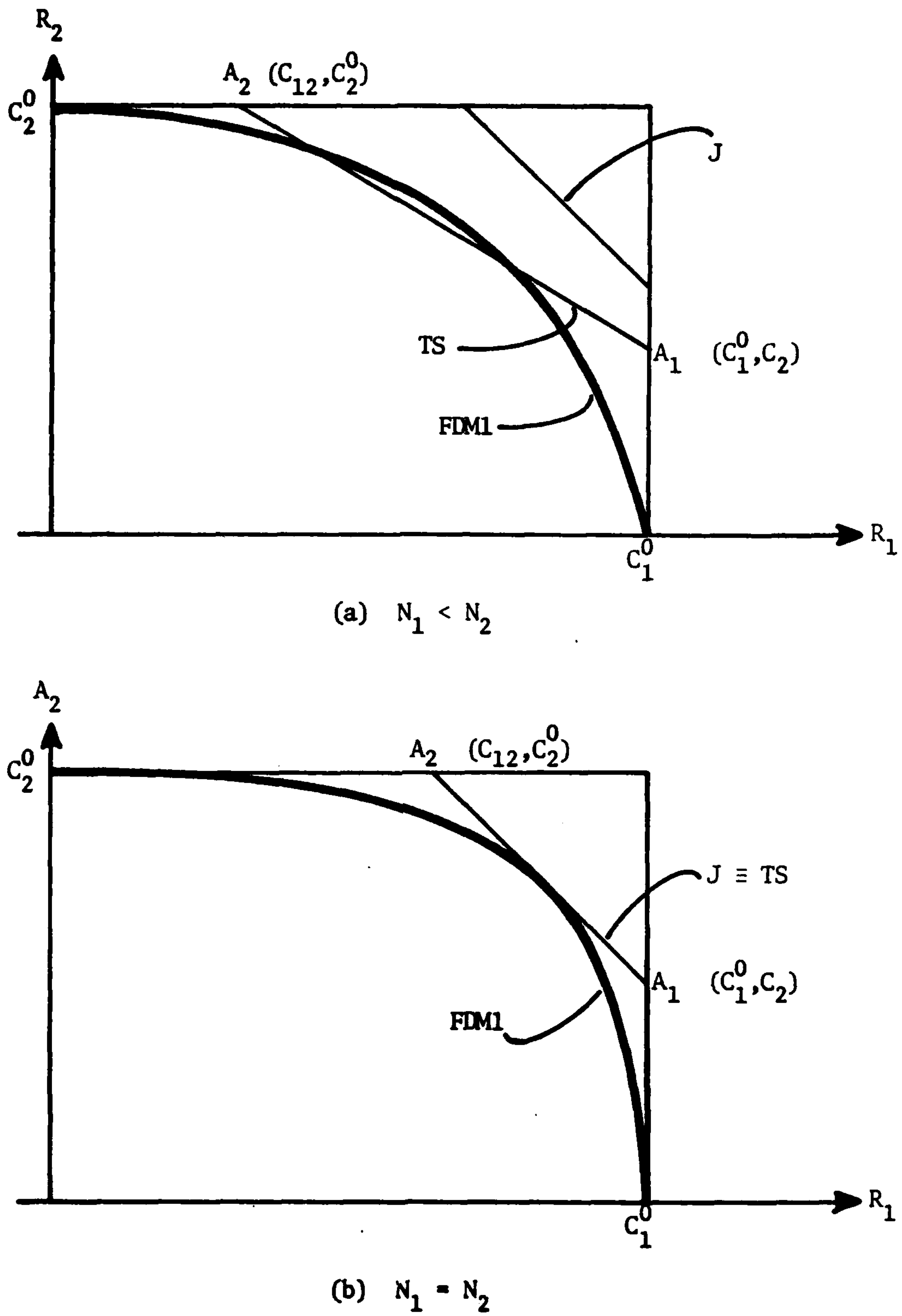


Figure 2

SKETCH OF SOME BOUNDS TO THE CAPACITY REGION

$$R_1 = \lambda W \log\left(1 + \frac{P_1/\lambda}{WN_1}\right)$$
$$R_2 = \bar{\lambda} W \log\left(1 + \frac{P_2/\bar{\lambda}}{WN_2}\right) \qquad (6)$$

Here we again consider a sufficiently long period of time and in a fraction λ of the time only x_1 is sent with power P_1/λ and in the remaining time only x_2 is sent with power $P_2/\bar{\lambda}$. We can show that the same rate pair can be achieved by frequency division also (let's call this a FDM1), where the frequency band is divided into two bands with bandwidths γW and $\bar{\gamma} W$ ($0 \leq \gamma \leq 1$, $\bar{\gamma} = 1-\gamma$) and the first band is used only for transmission of x_1 and the second only for x_2. Accordingly we have

$$R_1 = \gamma W \log\left(1 + \frac{P_1}{\gamma W N_1}\right)$$
$$R_2 = \bar{\gamma} W \log\left(1 + \frac{P_2}{\bar{\gamma} W N_2}\right) \quad , \qquad (7)$$

which is easily seen to be equivalent to TDM1. The rate pair is $(C_1^0, 0)$ for $\gamma=1$ and $(0, C_2^0)$ for $\gamma=0$. The curve obtained by varying γ between 0 and 1 or the region in the first quadrant of the R_1-R_2 plane surrounded by the curve is called FDM1. It is easily seen that the curve FDM1 is concave as shown in Fig. 2(a).

In the previous paper, we made mention of these two achievable regions TS and FMD1 but did not compare the performance of these two regions further. The comparison will be performed in Section IV and we shall see that the relation of the two regions is of great importance for further development of this paper.

A general outer bound to the achievable region for the two-user channel was derived in the previous paper, and in this Gaussian two-user channel the bound was shown to be a part of the before-mentioned rectangular region that lies on the left-lower side of the straight line $R_1+R_2=C_0$, where C_0 is given by

$$C_0 = W\log\left(1+\frac{P_1+P_2}{WN_1}\right) \tag{8}$$

When $C_0 \leq C_1^0+C_2^0$ or equivalently when $P_1 \geq W(N_2-N_1)$ the straight line $R_1+R_2=C_0$ cut the rectangle, but when $C_0>C_1^0+C_2^0$ the straight line is outside of the rectangle. In the latter case the rectangle itself is an outer bound to the achievable region. We denote by J this outer bound or the part of the straight line inside the rectangle.

In the previous paper we showed that when the noise power N_1 and N_2 are equal this two-user channel becomes equivalent to a multiple access channel and both our achievable region TS and outer bound J coincide and are equal to the capacity region and that the FDM1 curve becomes tangential to a line $\overline{A_1A_2}$ (or $R_1+R_2=C_0$). TS and J coincide only if $N_1=N_2$. The behavior is sketched in Fig. 2(b).

III. A NEW OUTER BOUND

In this section we show that the tighter outer bound than J obtained in the previous paper can be derived by simply utilizing the known capacity region of the corresponding Gaussian broadcast channel.

We first note that the corresponding problem of Gaussian broadcast channel is quite the same with that of the Gaussian two-user channel except for the condition imposed on powers of two signals. That is, for the broadcast channel, only the total power P' is allotted and the powers of the respective transmitters P_1', P_2' may be varied freely under the restriction of constant total power:

$$P_1' + P_2' = P' , \tag{9}$$

whereas for our two-user channel the powers of each transmitter P_1 and P_2 are allotted separately and both are kept constant.

Next, an exact capacity region of the Gaussian broadcast channel [2] and its boundary curve is given by

$$\begin{aligned} R_1 &= W \log \left(1 + \frac{P_1'}{WN_1}\right) , \\ R_2 &= W \log \left(1 + \frac{P_2'}{WN_2 + P_1'}\right) , \end{aligned} \tag{10}$$

where P_1' and P_2' are varied under restriction (9).

Now we consider our Gaussian two-user channel with powers P_1 and P_2. Let the sum of these powers be P:

$$P \equiv P_1 + P_2 . \tag{11}$$

Then we can say that the capacity region of the corresponding broadcast channel whose P_1' and P_2' in (10) should be varied under $P_1'+P_2'=P(=P_1+P_2)$ must be an outer

bound to our two-user problem since the broadcast channel problem is more general than ours and includes our problem as a special case when $P_1'=P_1$ and $P_2'=P_2$. The curve (10) under $P_1'+P_2'=P$ of course passes through $A_1(C_1^0,C_2)$ for $P_1'=P_1$ and $P_2'=P_2$. This curve also passes through the point $(C_0,0)$ where C_0 is given by (8). A straight line $R_1+R_2=C_0$ that often comprises the boundary of the outer bound J also passes through the same point. It can then be easily shown that the curve (10) is convex:

$$\frac{d^2R_2}{dR_1^2} \leq 0$$

and the derivative is negative and its absolute value is not greater than 1:

$$D(\lambda) = -\frac{dR_2}{dR_1} = \frac{\lambda P + WN_1}{\lambda P + WN_2} \leq 1 \quad , \tag{12}$$

where $\lambda=P_1'/P$. From the above arguments we can conclude that this curve is below the straight line $R_1+R_2=C_0$ in the first quadrant and its behavior is sketched in Fig. 3. Because the rectangle surrounded by $R_1=C_1^0$, $R_2=C_2^0$, R_1-axis and R_2-axis was also shown to be an outer bound in the previous paper, a part of the rectangle on the left-lower side of the above curve has been shown to be a new tighter outer bound. Lets call this new outer bound J'. We sometimes also call the part of the boundary curve (10) within the rectangle as J'.

One important conclusion obtained from the above result is that the point A_1 is an optimal achievable point on the line $R_1=C_1^0$ because the point A_1 is achievable and the outer bound J' passes through A_1.

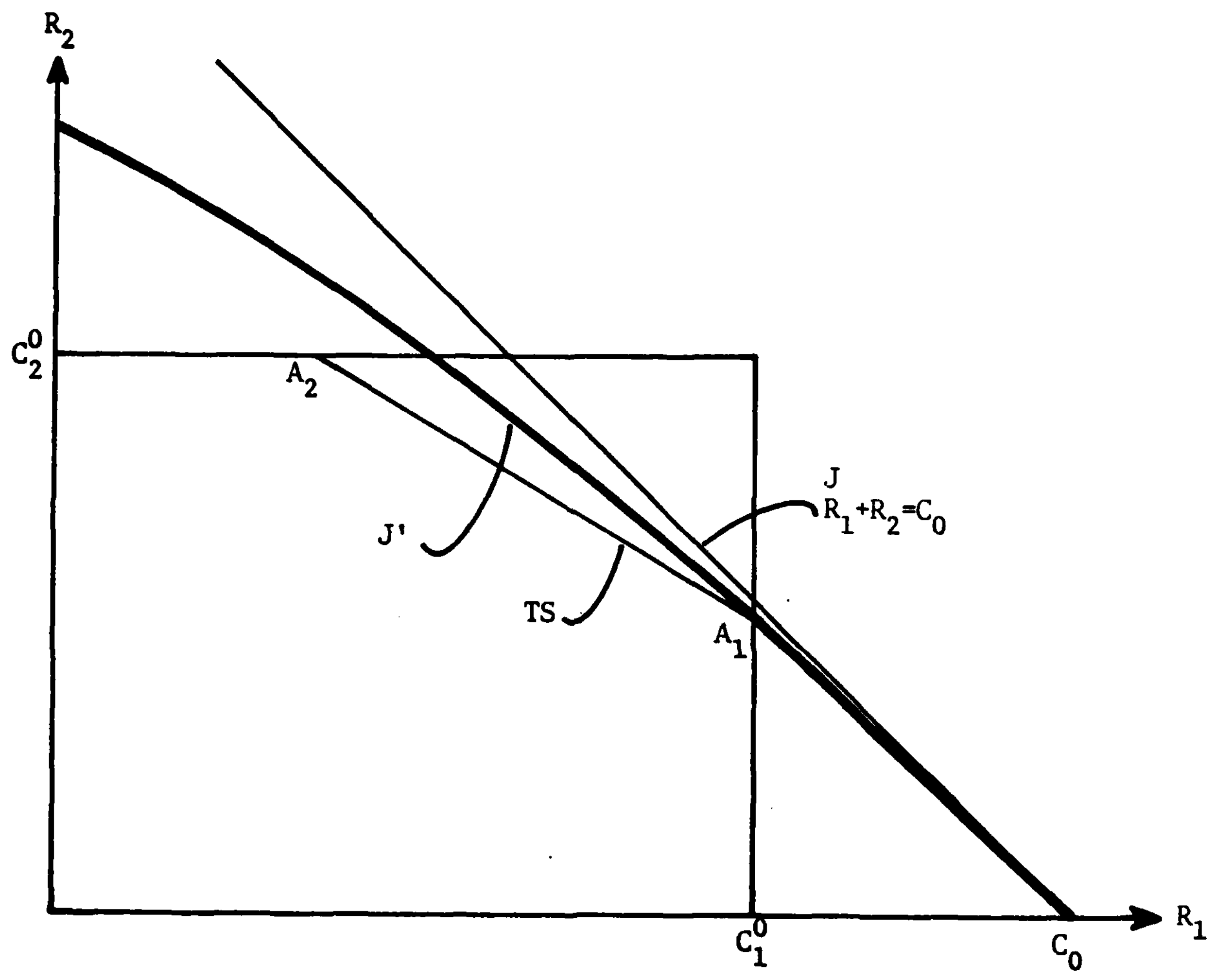


Figure 3

OUTER BOUNDS J AND J'

In Fig. 4 and 5 we show the achievable regions TS and FDM1 and old and new outer bounds J and J'. The abscissa and the ordinate are the rates per one degree of freedom, that is, $x=R_1/2W$, $y=R_2/2W$; and $S_1=S_2=5$ in Fig. 4 and $S_1=1$, $S_2=0.2$ in Fig. 5, where parameters S_1 and S_2 are defined by

$$S_1 = \frac{P_1}{WN_1}, \qquad S_2 = \frac{P_2}{WN_2} \tag{13}$$

and another parameter K is defined by

$$K = N_1/N_2 \quad (\leq 1) . \tag{14}$$

FDM1 depends only on S_1 and S_2 and not on K as seen in (7), and the point A_3 where J' crosses the upper side of the rectangle also does not depend on K. The x coordinate x_3 of the point A_3 is easily shown to be

$$x_3 = \frac{1}{2}\log\left(1+\frac{S_1}{1+S_2}\right) . \tag{15}$$

In Fig. 4, TS and FDM1 intersect each other only for K=.8 and for K=.05 the straight line $R_1+R_2=C_0$ is not shown since it is outside the rectangle. In Fig. 5, TS and FDM1 intersect each other for both K=0.8 and 0.2 and for K=0.2 the line $R_1+R_2=C_0$ is not shown since it is also outside the rectangle. In these figures, we can see that the new outer bound J' improves the old J fairly well and an inner bound TS or FDM1 already approaches fairly close to the outer bound J'.

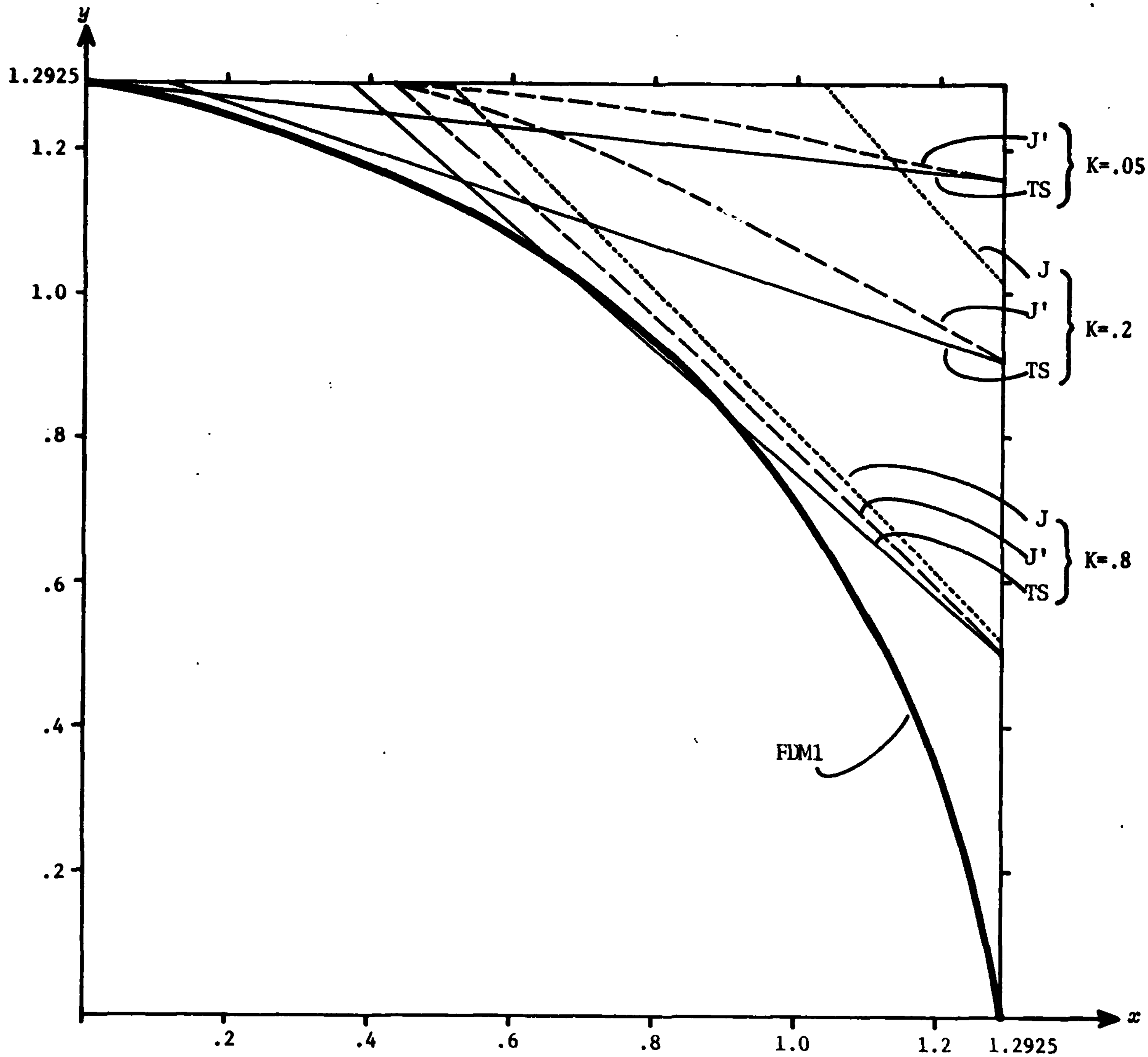


Figure 4 SOME BOUNDS FOR $S_1=S_2=5$

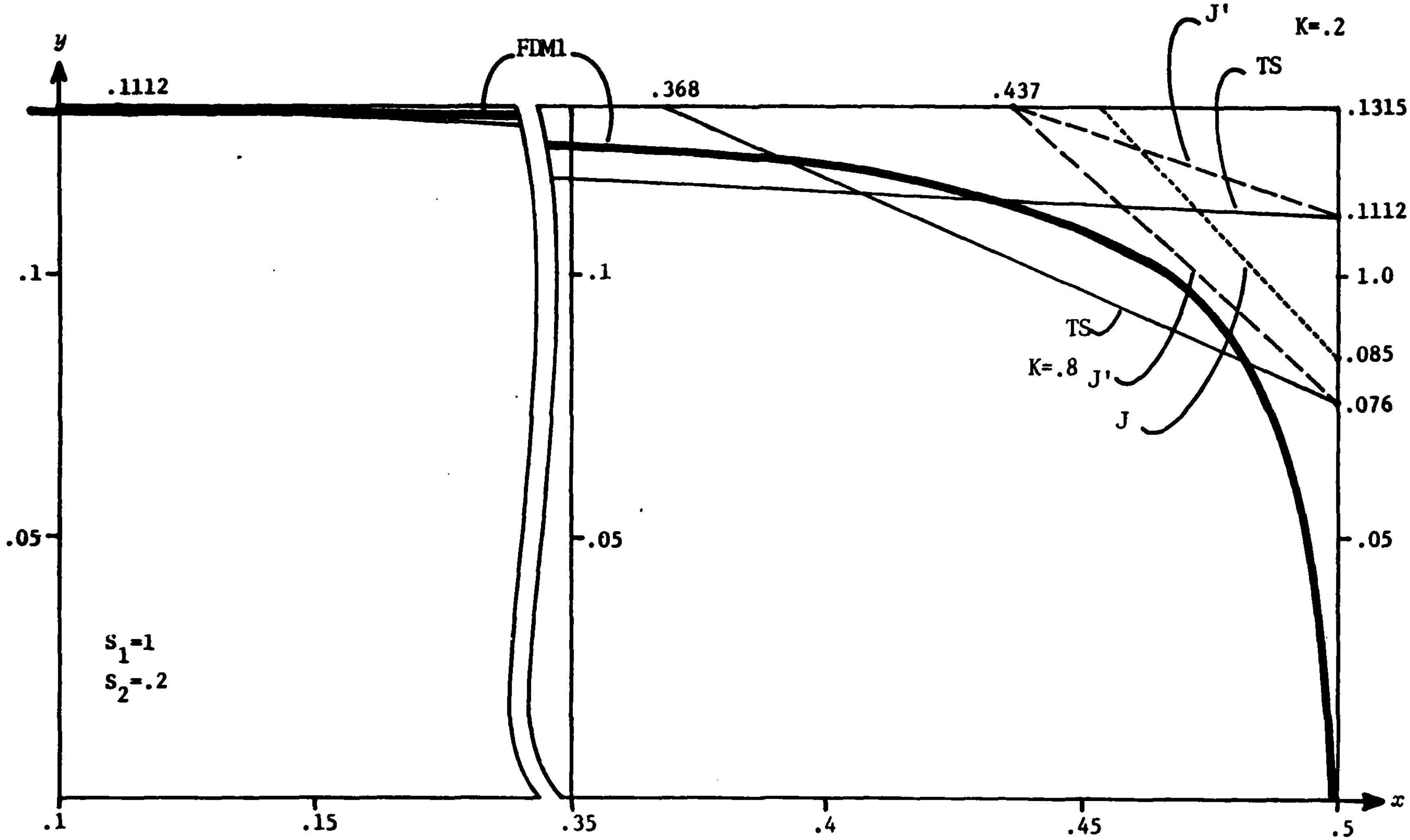


Figure 5

SOME BOUNDS FOR $S_1=1$, $S_2=.2$

IV. COMPARISON OF TWO ACHIEVABLE REGIONS TS AND FDM1

In the previous paper, we showed that the two superposition modes and therefore the region TS obtained by time sharing these two points are achievable. We also introduced a TDM1 (6) that is equivalent to FDM1 (7) as an achievable region, but we did not compare these two regions with one another. We cannot establish a general inclusion relation between these two regions but sometimes they intersect each other and sometimes the region TS includes FDM1. Since knowledge of the behavior of these two regions will be very important to understand the behavior of the further improved regions which will be discussed in succeeding sections, we shall discuss it in some detail here.

Fig. 6 is a graph showing when these two achievable regions intersect each other and when they do not. In this graph the abscissa is $S_1=P_1/WN_1$ measured in decibels and the ordinate is $S_2=P_2/WN_2$ also measured in decibels. Curves on which the boundary of FDM1 is just tangential to the TS straight line for three values of K, .8, .2 and .05 is also shown. For values of S_1 and S_2 that lie on the upper-left side of this curve, the FDM1 curve does not cross the TS straight line and therefore the region FDM1 is completely included within the region TS, and for S_1 and S_2 on the other side of the above curve FDM1 crosses with TS straight line at two points. As is seen in Fig. 6 these curves are nearly straight lines that pass near the origin with derivatives a little smaller than 1 and are shifted downward as K decreases.

Let us explain some typical behaviors of these two achievable regions for several S_1, S_2 values.

First we consider the case where TS includes FDM1 for all values of $K<1$. This case corresponds to the upper left region of Fig. 6 and the region is denoted by F in the figure. The graph of Fig. 7(a) shows a typical behavior

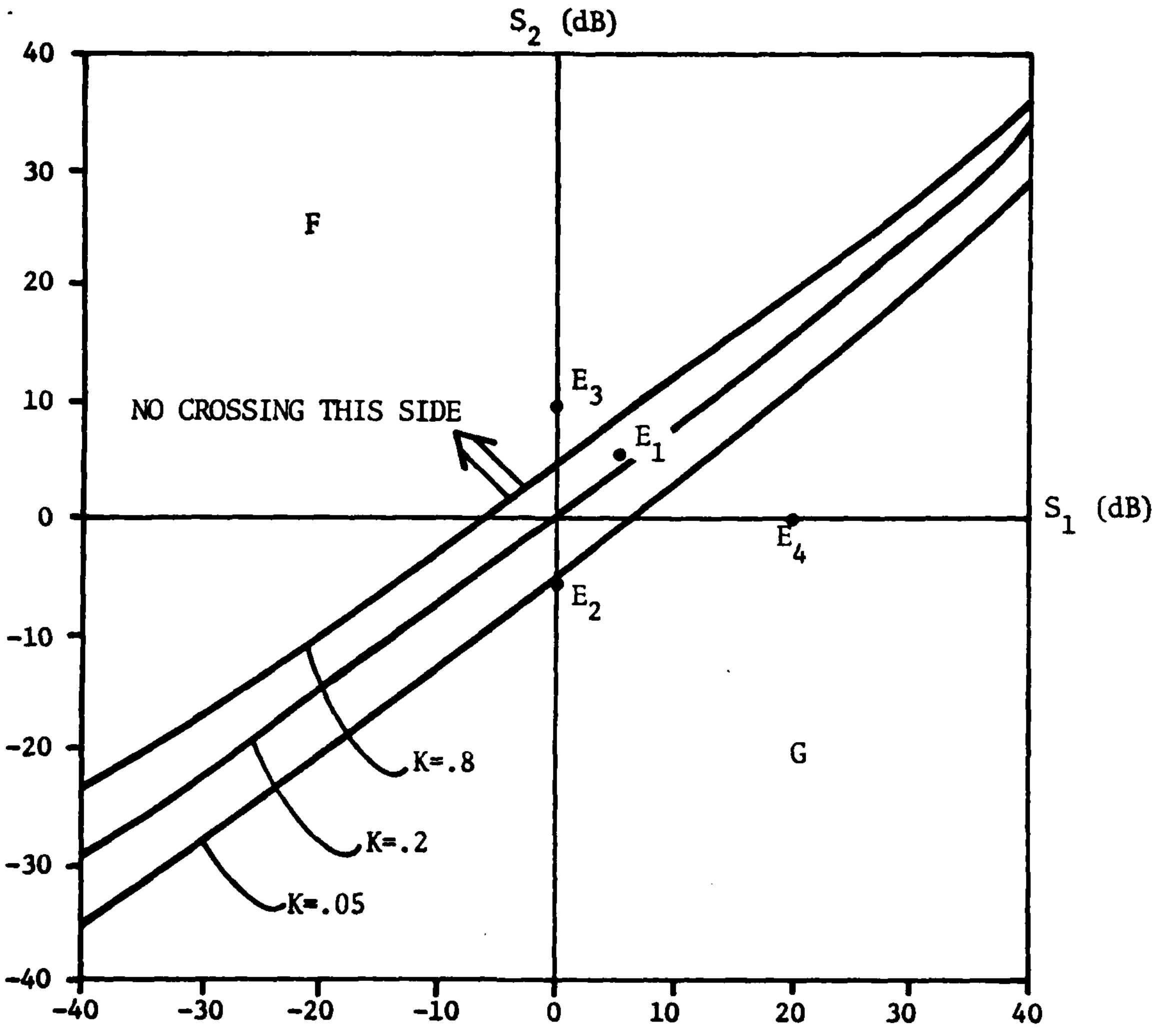


Figure 6

GRAPH SHOWING THE RELATION BETWEEN TS AND FDM1

of TS and FDM1 in this case for $S_1=1$ and $S_2=10$ that corresponds to the point E_3 in Fig. 6. Since $S_1<S_2$ holds in F, the rectangle in Fig. 7(a) is rather tall, and the point A_1 is located at a fairly high position, and A_1 moves upwards and A_2 moves leftwards as K decreases. The graph of Fig. 4 shows a similar behavior for $S_1=S_2=5$ that corresponds to the point E_1 in Fig. 6. In this case, two curves cross each other at adjacent two points only for large value of K, but they get separated and don't intersect each other for most values of K.

Next let us consider the region denoted by G in Fig. 6 where S_1 is very large and $S_2\leq 1$. In this region and when K is not small TS crosses with FDM1 only near the point A_1. The typical behavior is shown in Fig. 7(b) for $S_1=100$ and $S_2=1$ that corresponds to the point E_4 in Fig. 6.

An example where two curves intersect each other to a remarkable extent is shown in Fig. 5 for $S_1=1$ and $S_2=.2$ that corresponds to the point E_2 in Fig. 6. The region of rate pair where two curves cross each other moves from right to left as K decreases, because A_1 always moves upwards and A_2 always moves leftwards for decreasing K.

V. FREQUENCY DIVISION MULTIPLEXING USING TWO SUPERPOSITION MODES (FDM2)

In subsequent sections, we consider improving the achievable region over either one or both of two regions TS and FDM1 which have been studied in the foregoing sections. We shall try to improve the achievable region by dividing

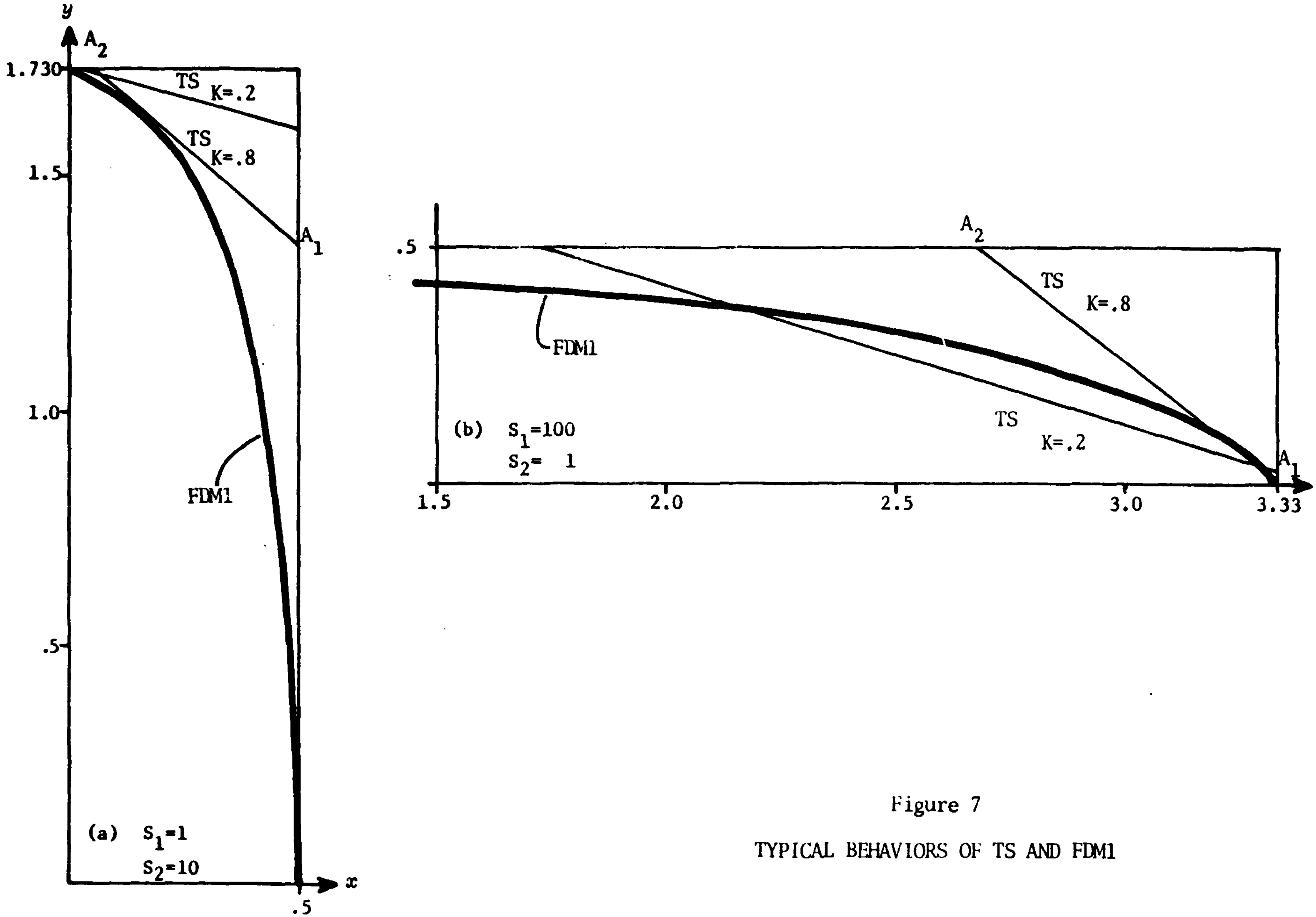


Figure 7

TYPICAL BEHAVIORS OF TS AND FDM1

the frequency band into two bands and utilizing superposition modes described in Section II in each sub-band. We consider in this section the case called FDM2 where two different superposition modes are used in two sub-bands and in the next section the case called FDM3 and FDM4 where the same modes are used in each sub-bands.

In FDM2, we divide the given frequency band W into two sub-bands of width γW and $\bar{\gamma}W$, where $0 \le \gamma \le 1$ and $\bar{\gamma}=1-\gamma$ as before. Signal $x_1(t)$ is also divided into two parts each within the respective frequency sub-bands and each with powers αP_1 and $\bar{\alpha}P_1$, where $0 \le \alpha \le 1$ and $\bar{\alpha}=1-\alpha$. Signal $x_2(t)$ is similarly divided into two parts with powers βP_2 and $\bar{\beta}P_2$, where $0 \le \beta \le 1$ and $\bar{\beta}=1-\beta$. The first band is operated by mode 1 and the second by mode 2 explained in Section II. We have therefore the following expression of achievable rates in FDM2:

$$\begin{aligned} R_1 &= \gamma W \log\left(1+\frac{\alpha P_1}{\gamma W N_1}\right) + \bar{\gamma} W \log\left(1+\frac{\bar{\alpha} P_1}{\beta P_2 + \bar{\gamma} W N_2}\right) \\ R_2 &= \gamma W \log\left(1+\frac{\beta P_2}{\alpha P_1 + \gamma W N_2}\right) + \bar{\gamma} W \log\left(1+\frac{\bar{\beta} P_2}{\bar{\gamma} W N_2}\right) \quad . \end{aligned} \tag{16}$$

Before discussing the properties of this FDM2, we note that the same can be done by time division multiplexing called TDM2. For that purpose, let's consider a sufficiently long period of time T and divide this into two parts λT and $\bar{\lambda}T$ in the similar way as we considered TDM1 in Section II. Let us utilize mode 1 during the first period of time using $x_1(t)$ and $x_2(t)$ with powers $(\alpha/\lambda)P_1$ and $(\beta/\lambda)P_2$ respectively, and utilize mode 2 during the remaining period of time using $x_1(t)$ and $x_2(t)$ with powers $(\bar{\alpha}/\bar{\lambda})P_1$ and $(\bar{\beta}/\bar{\lambda})P_2$. We can easily show that the powers averaged over all the periods of time take the required values P_1 and P_2 by the allocations of powers into two time intervals. We then have the folloiwng expression in TDM2 that is exactly equivalent to FDM2:

$$R_1 = \lambda W \log\left(1+\frac{(\alpha/\lambda)P_1}{WN_1}\right) + \bar{\lambda} W \log\left(1+\frac{(\bar{\alpha}/\bar{\lambda})P_1}{(\bar{\beta}/\bar{\lambda})P_2+WN_2}\right)$$

$$R_2 = \lambda W \log\left(1+\frac{(\beta/\lambda)P_2}{(\alpha/\lambda)P_1+WN_2}\right) + \bar{\lambda} W \log\left(1+\frac{(\bar{\beta}/\bar{\lambda})P_2}{WN_2}\right) . \tag{17}$$

Now we explain three properties of the region FDM2 obtained by varying the three parameters α, β, γ each between 0 and 1.

<u>Property VA: The region TS is included in FDM2.</u> This can be easily shown by letting both α and β equal to γ in (16). Then we just obtain the expression (5) for TS.

<u>Property VB: The region FDM1 is included in FDM2</u>. This is also seen easily by letting $\alpha=1$ and $\beta=0$ in (16). Then we obtain the expression (7) for FDM1.

The next property concerns the behavior of FDM2 when $N_1=N_2=N$. It should become equal to the capacity region since TS does as explained in Section II.

<u>Property VC: For $N_1=N_2=N$, the region FDM2 is equal to the known capacity region</u>. This can be shown by adding R_1 and R_2 in (16) after letting $N_1=N_2=N$, and then by letting the parameters satisfy $(\alpha-\gamma)P_1+(\beta-\gamma)P_2$ or more specially $\alpha=\beta=\gamma$. Then we can realize the entire portion of the straight line $R_1+R_2=C_0=W\log[1+(P_1+P_2)/WN]$ within the rectangular region by varying $\alpha=\beta=\gamma$ from 0 to 1.

We show a sketch of many achievable regions in Fig. 8. FDM2 is seen to include both TS and FDM1 due to the properties VA and VB described above.

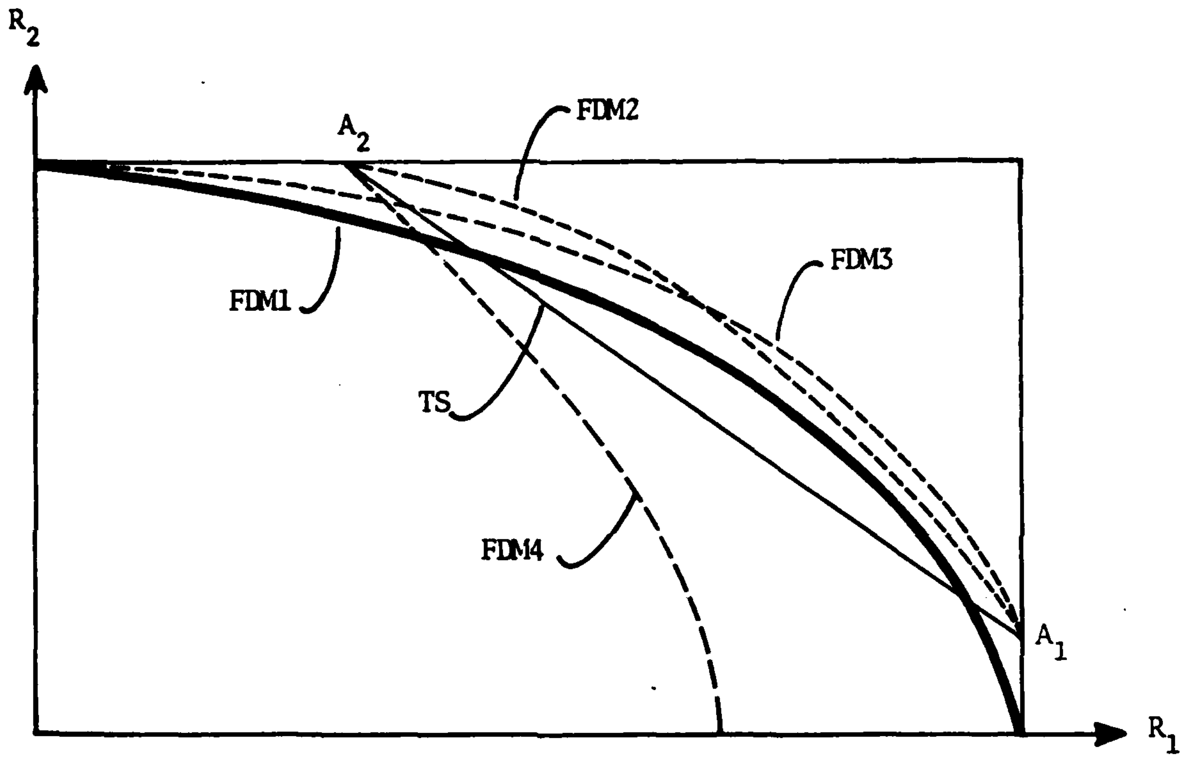


Figure 8

SKETCH OF MANY ACHIEVABLE REGIONS WHEN $N_1<N_2$

VI. FREQUENCY DIVISION MULTIPLEXING USING A SINGLE SUPERPOSITION MODE (FDM3)

In the same way as in FDM2, suppose we divide the given frequency band into two sub-bands of bandwidth γW and $\bar{\gamma} W$, let the parts of power of signal $x_1(t)$ in each sub-band be αP_1 and $\bar{\alpha} P_1$ and also let the parts of power of signal $x_2(t)$ in each sub-bands be βP_2 and $\bar{\beta} P_2$. While the two sub-bands were operated by different superposition modes in FDM2, we consider the case where both bands are operated by the same superposition mode. First let us utilize mode 1 for both sub-bands and call this FDM3. We have the following expression for achievable rates in FDM3:

$$\begin{aligned} R_1 &= \gamma W \log\left(1 + \frac{\alpha P_1}{\gamma W N_1}\right) + \bar{\gamma} W \log\left(1 + \frac{\bar{\alpha} P_1}{\bar{\gamma} W N_1}\right) \\ R_2 &= \gamma W \log\left(1 + \frac{\beta P_2}{\alpha P_1 + \gamma W N_2}\right) + \bar{\gamma} W \log\left(1 + \frac{\bar{\beta} P_2}{\bar{\alpha} P_1 + \bar{\gamma} W N_2}\right) \end{aligned} \tag{18}$$

It should be noticed here that we can of course consider the multiband system where the number of sub-bands is larger than two and each sub-band is operated by the same superposition mode. We shall, however, restrict ourselves to the two sub-bands system in this paper mainly because of the ease of analysis and numerical computation. However, many of the properties of FDM3 considered here might be valid in the multiband system.

Next we notice that we can again show the equivalence of the time division multiplexing to the frequency division multiplexing for this case. Since the proof is quite similar to that in the previous section we will not repeat it here.

We now discuss several properties of the region FDM3 obtained by varying the three parameters α, β, γ each between 0 and 1.

Property VIA: The region FDM1 is included in FDM3. This can be easily shown by putting $\alpha=1$, $\beta=0$ or $\alpha=0$, $\beta=1$ in (18).

Property VIB: A rate pair (C_1^0, C_2) corresponding to a point A_1 is realized iff $\alpha=\beta=\gamma$. C_2^0, the maximum of R_2, is realized iff $\gamma=\bar{\alpha}=\beta=0$ or $\bar{\gamma}=\alpha=\bar{\beta}=0$, and then $R_1=0$. The proofs are not difficult and will not be shown here.

Property VIC: For fixed γ and α (then R_1 is fixed) R_2 takes its maximum value for the following value of β:

$$\begin{aligned} \beta &= \beta_0 && \text{if} \quad 0 \leq \beta_0 \leq 1 \\ &= 1 && \text{if} \quad \beta_0 > 1 \\ &= 0 && \text{if} \quad \beta_0 < 0 \ , \end{aligned}$$

where

$$\beta_0 = \{(P_1+P_2)\gamma - P_1\alpha\}/P_2 \ .$$

The proof is straightforward and omitted. This property is very useful for numerical calculations of FDM3 because the number of parameters that should be varied can be decreased from 3 to 2.

Property VID: When $N_1=N_2=N$, a portion $\overline{A_1A_4}$ of the boundary line segment $\overline{A_1A_2}$ of the capacity region is realized by FDM3, where coordinates of the point A_4 are given by

$$A_4 \qquad (P_1C_0/(P_1+P_2),\ P_2C_0/(P_1+P_2)) \ .$$

It is noted that FDM1 is tangential to $\overline{A_1A_2}$ at A_3 for $N_1=N_2=N$. In order to prove the above property, we first add R_1 and R_2

$$R_1 + R_2 = \gamma W \log(1+\frac{\alpha P_1+\beta P_2}{\gamma WN}) + \bar{\gamma} W \log(1+\frac{\bar{\alpha} P_1+\bar{\beta} P_2}{\bar{\gamma} WN}) \ .$$

Then by differentiating it by α and β we see R_1+R_2 can take its maximum value

$$C_0 = W \log(1+\frac{P_1+P_2}{WN})$$

for α and β satisfying

$$(\alpha-\gamma)P_1 + (\beta-\gamma)P_2 = 0 \ .$$

In Fig. 8 we also show a sketch of the boundary curve FDM3. By the property VIA, it should be outside of FDM1 curve.

Next let us consider the case where both of the two sub-bands are operated by superposition mode 2 and call this FDM4. Then we have the following expression for achievalbe rates

$$\begin{aligned} R_1 &= \gamma W \log(1+\frac{\alpha P_1}{\beta P_2+\gamma WN_2}) + \bar{\gamma} W \log(1+\frac{\bar{\alpha} P_1}{\bar{\beta} P_2+\bar{\gamma} WN_2}) \\ R_2 &= \gamma W \log(1+\frac{\beta P_2}{\gamma WN_2}) + \bar{\gamma} W \log(1+\frac{\bar{\beta} P_2}{\bar{\gamma} WN_2}) \end{aligned} \tag{19}$$

where parameters α, β, γ have the same meaning as FDM3.

While FDM3 starts from point A_1 because it utilizes superposition mode 1, FDM4 starts from point A_2 because it utilizes mode 2. But we can see that FDM4 is included within the region TS and therefore useless in the improvement of the achievable region. In order to show this, we first add R_1 and R_2 of (19) and obtain

$$R_1 + R_2 = \gamma W \log(1 + \frac{\alpha P_1 + \beta P_2}{\gamma W N_2}) + \bar{\gamma} W \log(1 + \frac{\bar{\alpha} P_1 + \bar{\beta} P_2}{\bar{\gamma} W N_2}) .$$

This equation is quite similar to that used to prove property VID, and therefore, by differentiating it by α and β, we can show that R_1+R_2 takes its maximum value

$$C_0' = W \log(1 + \frac{P_1+P_2}{WN_2})$$

when α and β satisfy the relation

$$(\alpha-\gamma)P_1 + (\beta-\gamma)P_2 = 0 .$$

Therefore the rate pair within FDM4 satisfies the inequality

$$R_1 + R_2 \leq C_0' ,$$

while the points on the TS line satisfies

$$R_1 + R_2 \geq C_0'$$

because the TS line passes through point A_2 for which $R_1+R_2=C_0'$ holds and the absolute value of the derivative of the TS line is smaller than 1. A typical behavior

of FDM4 region is also sketched in Fig. 8. Although we will not mention FDM4 further since it does not improve the achievable region, we note here one property of FDM4 for $N_1=N_2=N$. We stated in Property VID that when $N_1=N_2=N$ a portion $\overline{A_1A_3}$ of the boundary line $\overline{A_1A_2}$ of the capacity region is realized by FDM3. We can show in a similar way that the remaining portion $\overline{A_2A_3}$ of $\overline{A_1A_2}$ is realized by FDM4 for $N_1=N_2=N$. Thus for $N_1=N_2=N$ we can attain the capacity region by utilizing both FDM3 and FDM4.

VII. RESULTS OF CALCULATIONS ON FDM2 AND FDM3 AND DISCUSSION

A sketch of FDM2 and FDM3 is shown in Fig. 8. Now we show in Fig. 9 and 10 the results of calculations of FDM2 and FDM3 for $S_1=S_2=5$ and for $S_1=1$ and $S_2=.2$ respectively. These two cases correspond to the points E_1 and E_2 in Fig. 6 and already many bounds were depicted for K=.8, .2 and .05 in Fig. 4 for the first case and for K=.8 and .2 in Fig. 5 for the second case. The results in Fig. 9 and 10 might have some errors in numerical calculations especially for FDM2 because we must change three parameters for it. The results however are enough to see the general trend of their behavior.

Improvements are so small in the first case that the results are shown in Fig. 9(a), (b) and (c) for K=.8, .2 and .05 respectively by expanding the scale of the ordinate. In these figures ordinate Δy represents the improvement of $y=R_2/2W$ over TS line. A part of the new outer bound J' is also shown and FDM1 is shown in Fig. 9(a) where FDM1 gets out of TS only in a small range

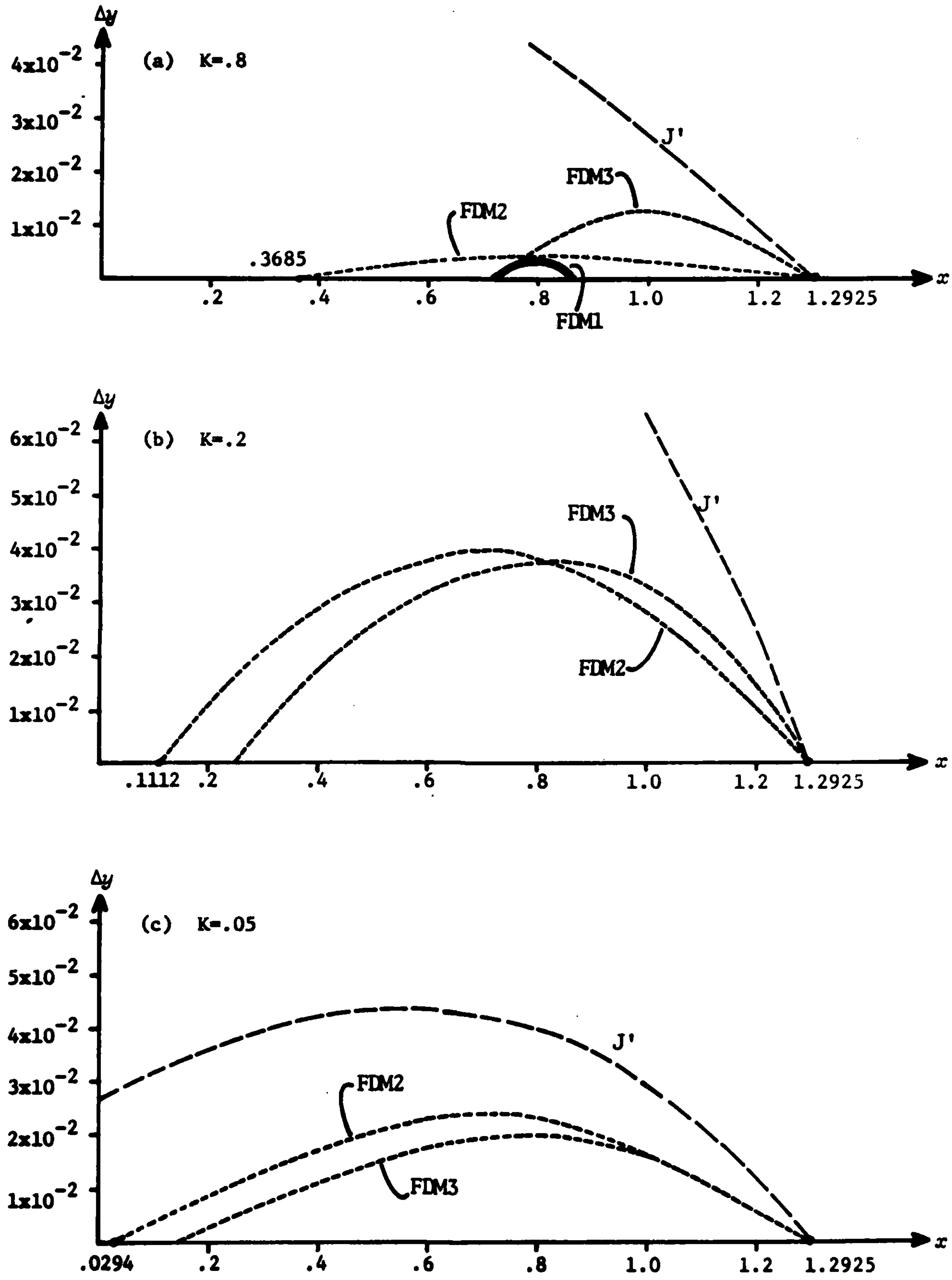


Figure 9

FDM2 AND FDM3 FOR $S_1 = S_2 = 5$

near x=.8. We can see from these figures that FDM2 is best near A_2 and FDM3 is best near A_1 although both are equally good near A_1 in (c) when K=.05.

In Fig. 10(a), (b) FDM2 and FDM3 are shown with some other bounds for K=.8 and .2 respectively. In these two figures it can be seen that there is some range between A_1 and A_2 where FDM1 is just as good as FDM2 and FDM3, and FDM2 is best to the left of it until A_2 and both FDM2 and FDM3 seem to be best to the right of it until A_1. In reality FDM3 is a little better than FDM2, but the difference is so small that we cannot see it in the figures in this case. We can see further that the boundary curves of FDM2 and FDM3 to the left and to the right of the middle range respectively is fairly close to the straight lines drawn from points A_2 and A_1 respectively so that the lines should be tangential to the FDM1 curve.

When we look through the results of calculations for many cases including the above two cases shown in Fig. 9 and 10, we can say that the boundaries of both regions FDM2 and FDM3 have a tendency to move being pulled by that of FDM1 when values of the parameters change. That is, these two regions which we are studying get far out of TS line if FDM1 does the same. On the contrary the two regions improve TS very little when FDM1 does not cross the TS line or gets a little out of TS line. This is partly explained by the fact that the region FDM1 is always included within both FDM2 and FDM3 as shown in the preceding sections.

From the above property of FDM2 and FDM3, it becomes appropriate to discuss their behavior by using the relation between TS and FDM1 stated in Section IV. There we stated that in region F of Fig. 6 where $S_1<S_2$ holds TS includes FDM1 and in the region G where S_1 is very large and $S_2 \leq 1$, FDM1 crosses TS very near to point A_1 when K is not so small. Let us denote by H other cases where TS and FDM1 cross each other in a remarkable way, and state

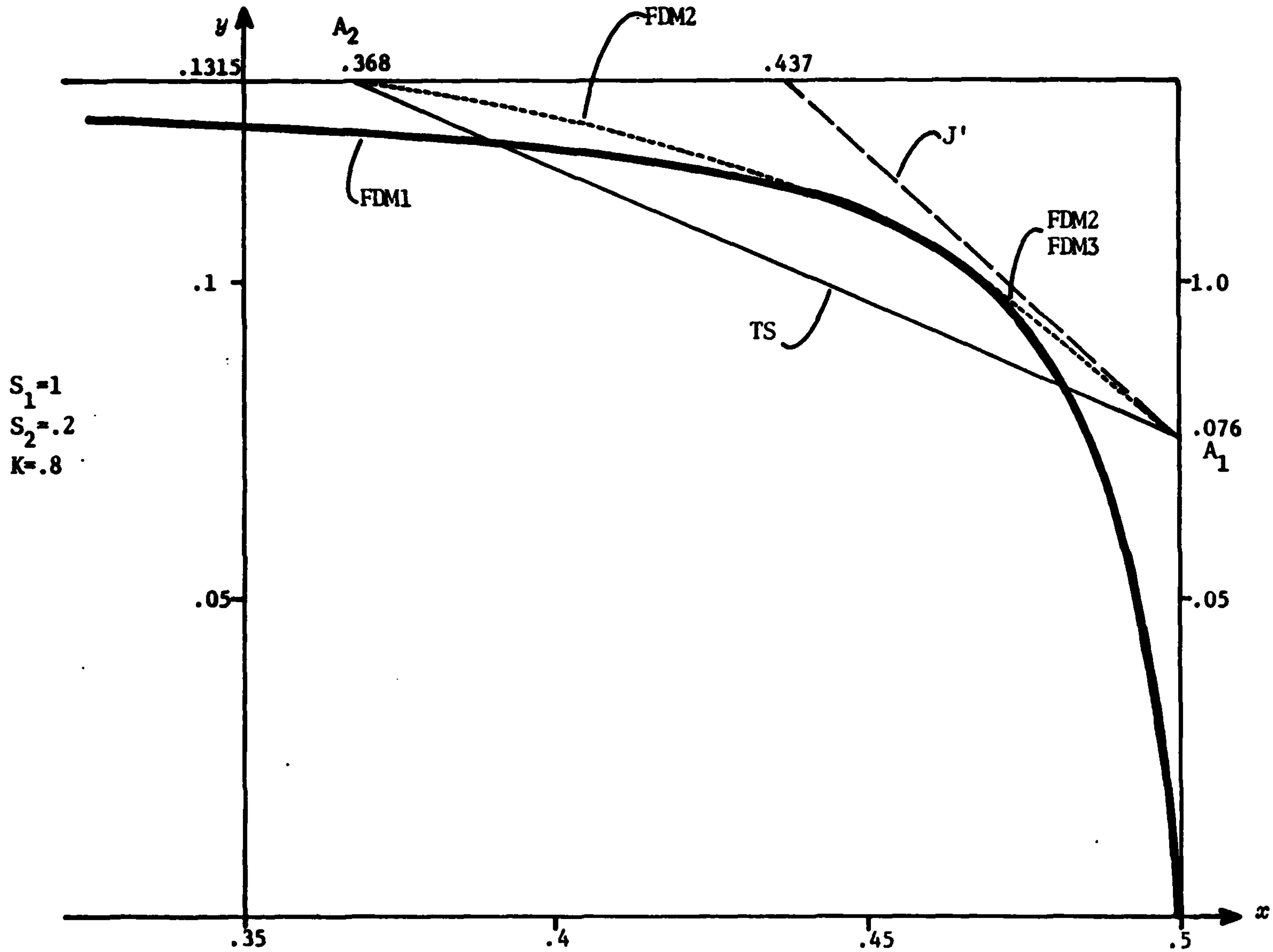
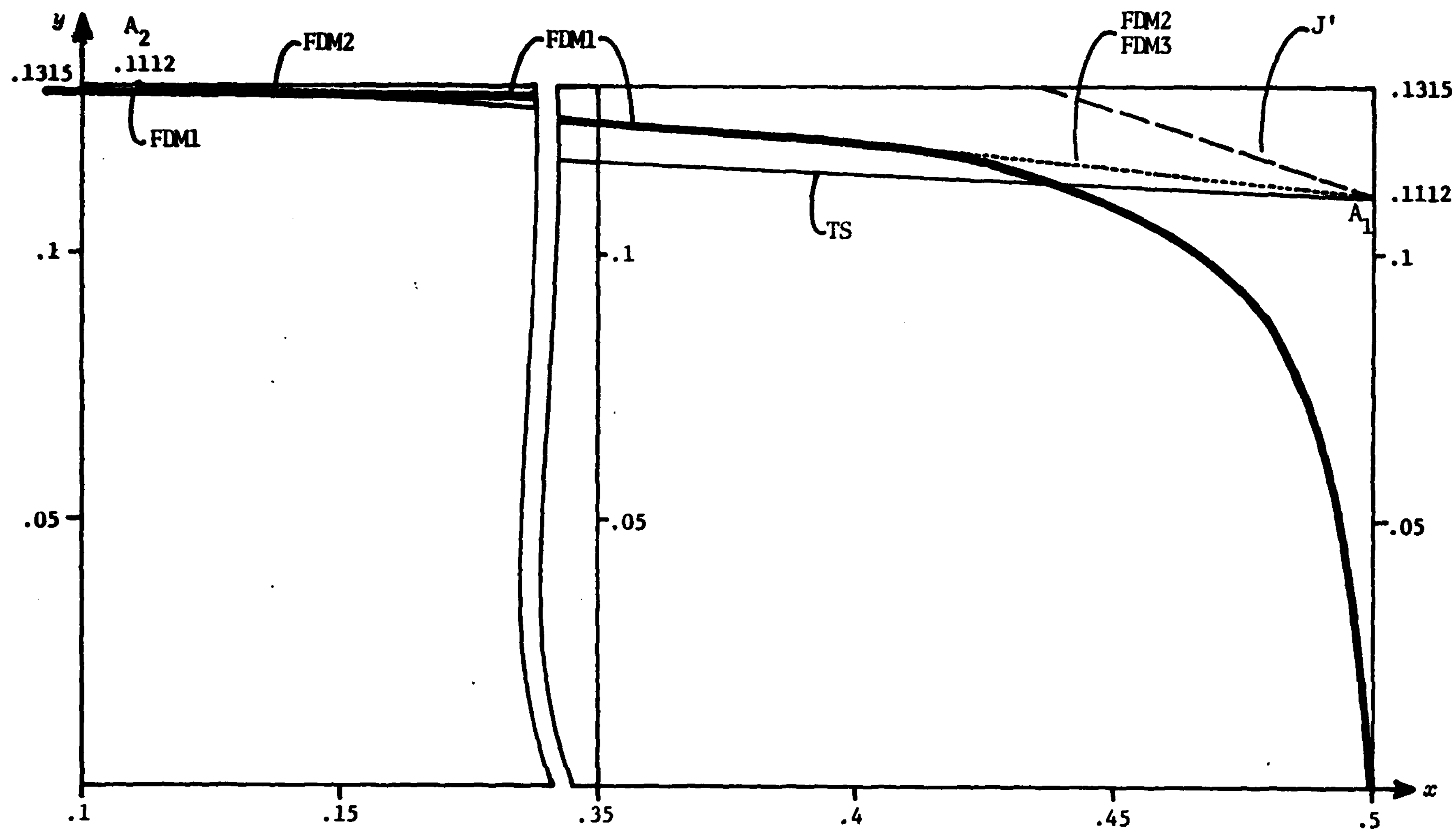


Figure 10(a)

FDM2 AND FDM3 FOR $S_1 = 1$, $S_2 = .2$ AND $K = .8$



$S_1=1$
$S_2=.2$
$K=.2$

Figure 10(b)

FDM2 AND FDM3 FOR $S_1 = 1$, $S_2 = .2$ AND $K = .2$

the behavior of FDM2 and FDM3 for the above three cases.

In the region F, improvement over TS is fairly small as in cases shown in Fig. 9 and FDM2 is better than FDM3 for all or for a very wide range of x. Sometimes FDM3 is best in the range very near to A_1.

In the region G for not so small K, the situation is very much similar to the above case, but FDM3 is always best in the small range near A_1.

For the other case H where two regions intersect each other in a remarkable way, there is almost always some middle range of x where FDM1 is as good as FDM2 and FDM3, and FDM2 is best to the left of it until A_2 and FDM3 is best to the right of it until A_1. Further, the following property seems to hold in general for the case H: The boundary curve of FDM2 on the left side of the middle range is close to the straight line drawn from A_2 so that this line should be tangential to the FDM1 curve, and the boundary curve of FDM3 on the right side of the middle range is sometimes also close to the straight line drawn from A_1 tangential to FDM1 as in the example shown in Fig. 10 and is sometimes fairly better than that.

By the properties of these achievable regions FDM2 and FDM3, we see that the following region should be a useful first approximation to the best achievable region known so far: For F and G (K should be not so small for G), take TS itself as a good approximation, and for H draw tangential straight lines from A_1 and A_2 to FDM1. The obtained region is a good first approximation.

So far in this paper, we have considered two achievable regions FDM2 and FDM3 to improve the regions TS and FDM1 introduced in the previous paper. It might be possible to further improve the achievable region by combining FDM2 and FDM3 or by adopting multiband frequency division where the number of sub-bands is larger than two. It is surmised by the author however that we have already achieved fairly well near A_1 because the point A_1 itself is optimum on the line $R_1=C_1^0$, but that some improvement might be

expected near A_2 by some unknown new methods.

VIII. CONCLUSION

We have studied the Gaussian two-user channel. It is a channel with two continuous inputs with allotted powers and with two continuous outputs where information should be transmitted between two input/output pairs simultaneously and where the two input signals are simply added and each output signal is a superposition of this summed signal with a different white Gaussian noise. In this paper the concept of two superposition modes introduced in the previous paper is first reviewed.

The achievable region TS, obtained by simply time-sharing these two superposition modes, and the one FDM1, obtained by transmitting only one and different input signal in each two divided frequency sub-bands are also reviewed and compared with each other.

A new, tighter outer bound than the one introduced in the previous paper for general two-user channels is obtained for this Gaussian two-user channel utilizing the known exact capacity region of the corresponding Gaussian broadcast channel.

Two kinds of frequency division multiplexing FDM2 and FDM3 are introduced and their properties are studied. In either case the given frequency band is divided into two sub-bands and these two sub-bands are operated by superposition mode 1 and 2 respectively in FDM2 and both sub-bands are operated by the superposition mode 1 in FDM3. The achievability of these two methods for

different parameter values is discussed.

It is pointed out that the achievable region, that is finally obtained in this way, is fairly well approximated by using the regions TS and FDM1.

REFERENCES

[1] SATO, H.: "Two-User Communication Channels," *ALOHA System Technical Report B75-29*, University of Hawaii, October 1975; also submitted to IEEE Trans. on Information Theory.

[2] BERGMANS, P.P. and COVER, T.M.: "Cooperative Broadcasting," *IEEE Trans. on Information Theory*, Vol. IT-20, No. 3, May 1974, pp. 317-324.

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professors N. Abramson, N.T. Gaarder and D. Slepian for their encouragements and criticisms.

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professors N. Abramson, N.T. Gaarder and D. Slepian for their encouragements and criticisms.